This may be shown by a transformation of variables. Let us adopt polar co-ordinates, the singular point being the pole, and thus put

$$s = \mathrm{w} \cos \theta + 2,$$
$$z = \mathrm{w} \sin \theta + 2.$$

The differential equation then becomes

$$\frac{d\mathrm{w}}{\mathrm{w}} = -\frac{\mathrm{w} \sin \theta \cos^2 \theta + \sin^2 \theta + \sin \theta \cos \theta}{\mathrm{w} \cos \theta \sin^2 \theta + 1 + \sin^2 \theta - \sin \theta \cos \theta} d\theta.$$

And we have

$$\int \frac{dz}{s - z} = \int \frac{d\theta}{\mathrm{w} \cos \theta \sin^2 \theta + 1 + \sin^2 \theta - \sin \theta \cos \theta}.$$

The denominator of these expressions cannot vanish unless w exceed 2, and it is plain that it remains positive and finite for all values of θ. Thus r becomes infinite only when θ does. Consequently there are an infinite number of solutions when $\rho_1/R = \frac{1}{3}$; and a less number when ρ_1/R is either less or greater than this. With the value we have attributed to this fraction in the case of the earth, the course of the curve shows that there is but one solution.

EXERCISES.

164

Derive geometrically the usual expressions for the radius of the circle inscribed in a triangle, and for the area of the triangle, in terms of the sides.

[*F. H. Loud.*]

165

In any triangle ABC let a circle be inscribed touching the sides AB, BC, CA in N, L, M respectively. Let the centre O of this circle be joined to the vertices, and from O let OP, OQ be drawn perpendicular respectively to OC, OB, and cutting BC in P and Q. Then if NP and AQ be drawn, these lines will be parallel as will also AP and MQ.

[*F. H. Loud.*]

166

A circle cuts a parabola and the centroid of the four points of intersection is found. What is the locus of the centre of the circle if this point be fixed?

[*W. M. Thornton.*]

167

If $y_1 y_2 = k^2$ (a constant), where (x_1, y_1) and (x_2, y_2) are points on the parabola $y^2 = 4ax$, find the locus of the intersections of the normals at these points.

[*R. H. Graves.*]

168

If the normals at four points on a rectangular hyperbola meet in a point, and the sum of the squares on the six distances between the four points, taken two together, is constant $(= k^2)$, prove that the locus of the point of concourse of the normals is a circle. [*R. H. Graves.*]

169

Find the locus of the instantaneous centre of a tangent to an ellipse when one point of the tangent moves in the auxiliary circle. [*R. H. Graves.*]

170

Prove that the circular projections on planes passing through one extremity of the transverse axis of an ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, of all sections through that axis, lie on a surface which is formed by eliminating φ between the two equations

$$(x - a\sin^2\varphi)^2 + (y - ac\lambda\sin\varphi)^2 + (z - ab\varkappa\sin\varphi)^2 = a^2\cos^2\varphi,$$

$$x\sin\varphi + c\lambda y + b\varkappa z = a\ \sin\ \varphi;$$

in which
$$\lambda = \left(\frac{b^2 - a^2\cos^2\varphi}{b^2 - c^2}\right)^{\frac{1}{2}},$$

and
$$\varkappa = \left(\frac{a^2\cos^2\varphi - c^2}{b^2 - c^2}\right)^{\frac{1}{2}}.$$

[*J. O'Byrne Croke.*]

171

A homogeneous heavy rod is hung from a fixed point by elastic threads of given length fastened at its extremities. Find the position of equilibrium.

[*W. M. Thornton.*]

172

Find the deflection of a homogeneous elastic beam of length $2v$, loaded uniformly, and supported at two points distant u from its middle point.

[*W. M. Thornton.*]

173

Find the deflection of a cross tie in a railway under the action of the driving

wheels of a locomotive, assuming the resistance of the road-bed to compression to be proportional to the deflection at the point considered. [*W. M. Thornton.*]

SELECTED.

174

FIND the locus of the intersection of the altitudes of a triangle whose base and area are given.

175

FIND the locus of the poles of normals to a given ellipse.

176

FIND the normal to an ellipse which cuts the curve again at the minimum angle.

177

Two conjugate diameters a', b' subtend at a point on the ellipse angles α', β'. Show that $\cot^2 \alpha' + \cot^2 \beta'$ is constant.

178

FIND the point in which the normal to $xy = m^2$ cuts the curve again.

179

FIND the area of the triangle formed by the asymptotes to an equilateral hyperbola and the normal to the curve.

180

FIND the envelope of a system of circles each of which is seen from two fixed points under a constant angle.

181

SHOW that the locus of the centres of equilateral hyperbolas circumscribed to a given triangle is the nine-points circle of the triangle.

182

THE centroid of the four points of intersection of a circle and an equilateral hyperbola, bisects the join of their centres.

183

IF $z = x + iy$ be a complex quantity whose geometric locus is a straight line, find the locus of $\zeta = z^2$.

184

SHOW how to express the area between the parabola $y = A + Bx + Cx^2$, the axes of co-ordinates, and the ordinate $x = 1$, by means of two given ordinates.

185

From a point taken on a fixed normal to a given ellipse the three other normals to the curve are drawn and the circumcircle of their feet is constructed. Find the locus of its centre.

186

Find the locus of points which may be the common centre of two similar conics, one circumscribed, the other inscribed to a given triangle.

187

On the diagonals of a complete quadrilateral three pairs of points are taken which divide these diagonals harmonically. Show that they form a Pascal's hexagon.

188

Three conics are bitangent each to the other two. Show that the chords of contact of two of them with the third, form with the common tangents to these two an harmonic pencil.

189

Show how to resolve a given force into three coplanar components acting in given lines not concurrent.

190

D, E, F are the feet of the altitudes of ABC. Find the resultant of the force represented by AE, AF, BF, BD, CD, CE.

191

A rod whose centroid is given, is hung from a smooth pin by a string fastened to its ends. Find the positions of equilibrium.

192

A frustum of a right circular cone stands on a rough inclined plane in a position of tottering equilibrium. Will it slide or topple over?

193

When will the centroid of a triangular frame of wire coincide with that of three particles P, Q, R at its angular points?

194

Show that the centroid of a hemispheroid of very small eccentricity e deviates from the middle points of its altitude by $\frac{23}{12} e^2$ thereof, nearly.

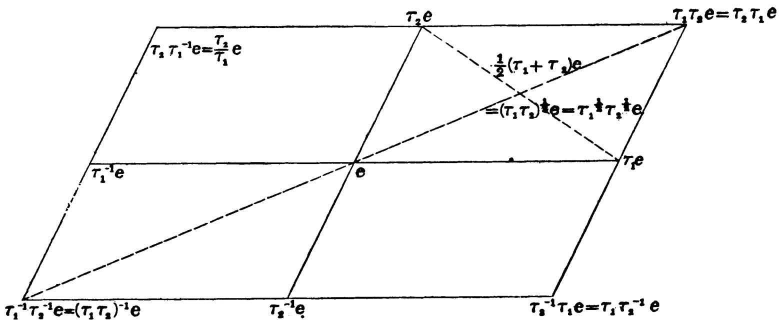

Fig. 1.

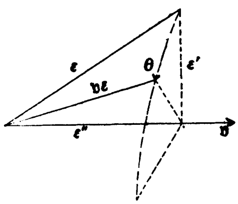

Fig. 2.

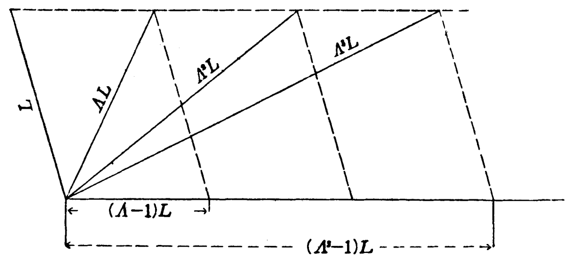

Fig. 4.

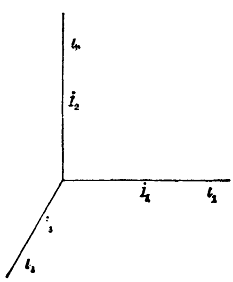

Fig. 3.

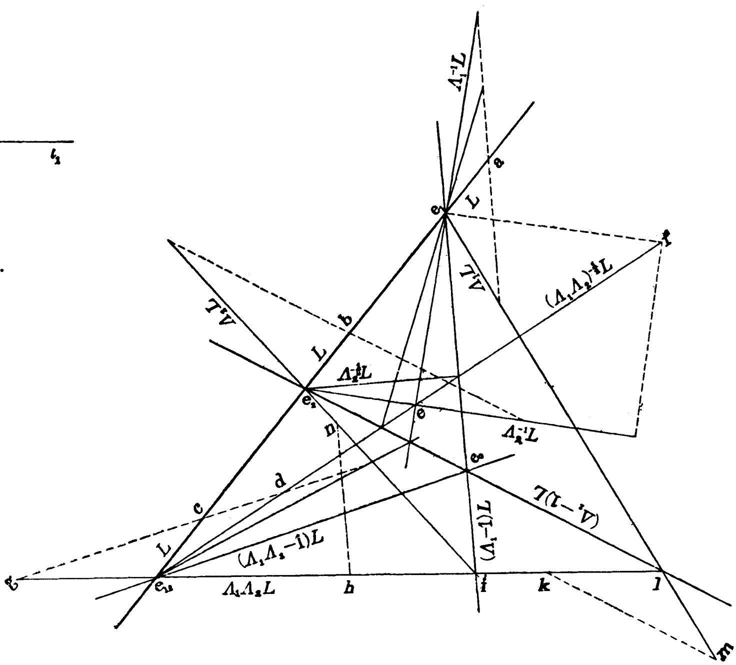

Fig. 5.